Chapter 1
A Match Made on Earth. On the Applicability of Mathematics in Physics

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Abstract Anyone interested in understanding the nature of modern physics will at some point encounter a problem that was popularized in the 1960s by the physicist Eugene Wigner: Why is it that mathematics is so effective and useful for describing, explaining and predicting the kinds of phenomena we are concerned with in the sciences? In this chapter, we will propose a phenomenological solution for this "problem" of the seemingly unreasonable effectiveness of mathematics in the physical sciences. In our view, the "problem" can only be solved—or made to evaporate—if we shift our attention away from the *why-question*—Why can mathematics play the role it does in physics?—, and focus on the *how-question* instead. Our question, then, is this: How is mathematics actually used in the practice of modern physics?

1.1 Introduction

Mathematics is everywhere. For some to their pleasure, for some to their agony and perhaps for some to their bafflement. We use it in every-day life as well as in the sciences. We use it as a tool of calculation and inference, and it also gives us a "deeper", more quantitative, more exact understanding of "how things really are". We use it to make predictions about the future of our universe or to trace things back to the past, as in the big bang model. Theoretical physicists sit at their desks and make quantitative predictions that later, sometimes decades later, experimentalists are able to verify. You open any textbook in engineering and science, from physics to economics, and you will encounter a plethora of mathematical

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symbols. The mathematics can be arithmetic, geometry, algebra, calculus, abstract algebra, linear algebra, topology, algebraic geometry and so on. In public discourse some theologians consider the applicability of mathematics even as strong evidence for the existence of God.

The philosophical problem surrounding the relation between mathematics and the empirical sciences is rather obvious: Why is it that mathematics is so effective and useful for describing, explaining and predicting the kinds of phenomena we are concerned with in the sciences? Philosophers, in their attempt to make sense of the enormous success of science, thus face what is commonly called the *applicability problem*, the problem of explaining the intimate tie between mathematics and science.

This problem is revived and reformulated by the physicist Eugene Wigner under the striking title of the *Unreasonable Effectiveness of Mathematics in the Natural Sciences* Wigner (1960). While Wigner mostly focuses on the case of modern theoretical physics, and its relationship with mathematics, he still fails to find a satisfactory solution for the applicability problem (Islami 2017). In a nutshell, Wigner's "solution" is that there is no solution: all we can say about the applicability problem is that "the enormous usefulness of mathematics in the natural sciences is something bordering on the mysterious" (Wigner 1960, 223). On Wigner's view,

> [t]he miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning. (Wigner 1960, 237)

Wigner is not the only one who thinks that the applicability problem cannot be solved. The physicist Paul Dirac echoes Wigner's remarks:

> [T]he mathematician plays a game in which he himself invents the rules while the physicist plays a game in which the rules are provided by Nature, but as time goes on it becomes increasingly evident that the rules which the mathematician finds interesting are the same as those which Nature has chosen. (Dirac 1939, 124)

And David Hilbert in a course lecture in (1919) says:

> We are confronted with the peculiar fact that matter seems to comply entirely to the formalism of mathematics. There arises an unforeseen harmony of being and thinking, which for now we have to accept like a miracle. (Hilbert 1992, 69; our translation)

The physicist David Gross takes the same line when he writes that it is "something of a miracle that we are able to devise theories that allow us to make incredibly precise predictions regarding physical phenomena" (Gross 1988, 8372).

1.2 The Applicability Problem(s)

The applicability problem, commonly viewed, concerns the relationship between mathematics and sciences, social as well as natural. Thus commentators on Wigner's applicability problem have argued for a range of positions from the Unreasonable Ineffectiveness of Mathematics in Economics (Velupillai 2005) to the Unreasonable Ineffectiveness of Mathematics in Biology (Lesk 2000, 29). The underlying assumption in treating applicability as a unified problem is a view of science as having a universal experimental method (as well as assigning a single method to mathematics itself). For us, however, modern physics presents a peculiar and especially interesting case of the applicability problem where mathematics plays a more fundamental role than being a mere language for the description of the physical phenomena. Following Wigner, then, we focus on the case of modern theoretical physics.

At the heart of applicability problem lies what we call the *distinctness thesis*, i.e. the thesis that mathematics and the physical sciences are categorically distinct. According to a widespread view, the challenge presented by the applicability problem is then to explain why, despite their distinctness, mathematics and physics are so closely intertwined. Yet, it is important to realize at this point that what the distinctness in question precisely amounts to can vary greatly depending on one's philosophical background assumptions.

Consider, for instance, a realist both with regard to mathematics and physics. A realist of this kind might be happy to follow Gödel in viewing mathematics as analogous to the physical sciences concerning their basic methodological outlook: All disciplinary differences notwithstanding, mathematics and physics are fundamentally similar in that both seek to describe a mind-independent reality that determines the truth-values of propositions in the respective areas (Gödel 1983, 456). However, even if mathematics and physics are viewed as analogous in this way, there is still an important sense in which they are categorically distinct: While the subject matter of physics is typically said to consist of concrete physical objects, mathematical realism is usually associated with the view that mathematical objects such as primes or polynomials exist outside of space and time, and independently of the causal relations in which concrete physical objects stand. And it is this *ontological distinctness* that gives rise to one specific version of the applicability problem: Why is it that knowledge about the *abstract* realm proves to be so enormously effective in generating knowledge about world of *concrete* physical phenomena? In order to solve this version of the applicability problem, a number of thinkers from the ancient Pythagoreans over 17$^{\text{th}}$ century scientists such as Kepler or Galileo to modern-day physicists like Max Tegmark have advocated some version of *mathematical monism*: Although our experiences tell us otherwise, it is argued that reality is ultimately *nothing but* mathematical structure. According to its proponents, the main advantage of this view is that it circumvents the ontological version of the distinctness thesis. And this, of course, also prevents the applicability problem from arising because "our successful theories are not mathematics approximating physics, but mathematics approximating mathematics" (Tegmark 2008, 125).

Given what has been said thus far, one might wonder how the distinctness thesis plays out under different philosophical background assumptions. Consider, for instance, a formalist who denies that mathematics should be viewed as a body of propositions with determinate truth-values, describing an abstract sector of reality. In order to avoid the metaphysical challenges posed by mathematical realism, formalists think of mathematics as a game-like endeavor in which strings of symbols are manipulated according to freely stipulated rules. Like chess pieces, the symbols with which the game of mathematics is played do not denote anything. The only meaning these symbols have is attributed to them by the mathematician who accepts certain arbitrary rules in order to participate in a game-like activity. Instead of being constrained by a theory-independent reality, the game of mathematics is only driven by inner-mathematical virtues such as rigor, elegance, simplicity, manipulability or formal beauty. For instance, Wigner defines mathematics in precisely this spirit, namely as "the science of skillful operations with concepts and rules invented for this purpose" (Wigner 1960, 2).

It is clear that for formalists, the applicability problem does not arise as the result of the *ontological* version of the distinctness thesis. Since, on their view, there is no abstract realm to begin with, there is also no ontological hiatus that would make the coordination between mathematics and physics appear particularly mysterious. This, however, does not mean that the applicability problem does not arise in a different form. It is easy to see why this is so: From a formalist perspective, mathematics is an arbitrary human creation which is only driven by genuinely inner-mathematical virtues. Physics, on the other hand, is first and foremost constrained by a basic commitment to empirical adequacy[1] and hence by the methodological principle that the ultimate guide in matters of theory-acceptance is adequacy with respect to the segment of reality a theory purports to describe.[2] It is this difference that gives rise to the *praxiological* (or *methodological*) distinctness thesis, and hence to a non-ontological version of the applicability problem: Why is it that a cognitive practice that is guided by virtues like rigor, elegance, simplicity, manipulability or formal beauty proves exceptionally successful in an area where it hard to see why these inner-mathematical virtues are epistemically relevant at all? Seen from this perspective, then, and building on a metaphor used by Wigner, our situation in physics is like that of a person who encounters a bunch of keys on display in an art exhibition. Although the artist who made the keys assures us that she had no practical

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[1] To be sure, it could be pointed out that criteria other than empirical adequacy do have their place in physics, especially when physicists invoke super-empirical virtues to break underdetermination on the level of empirically equivalent theories. However, apart from the fact that it is unclear if such super-empirical values should qualify as genuinely epistemic, the role of virtues like rigor, elegance, simplicity, manipulability or formal beauty seem much more fundamental in mathematical research. In mathematics, these virtues are not merely the means to decide between otherwise indistinguishable theories; they are rather the guiding principles for the development and assessment of theories.

[2] Of course, one could question whether such a "empirical paradigm of theory assessment" is adequate in light of more recent developments in contemporary physics. For instance, since the scale of string theory is roughly of the order of the Planck length, the chances of finding direct empirical confirmation of the theory's core claims seem rather remote. Given this lack of empirical backing, it is not surprising that string theorists are guided to a much stronger extent by considerations resembling those used in pure mathematics. However, since the jury is still out on whether string theory is in fact too far detached from the binding norms of experimental science or, alternatively, on how the research practice of string theorists would change if empirical tests were possible, we will ignore this case here (cf., for further discussions, Penrose 2004; Smolin 2006; Dawid 2013; Hossenfelder 2018).

purposes in mind, we find to our surprise that some of the keys unlock our doors at home. Confronted with this situation, two options seem to be available: either we admit that the situation is indeed highly mysterious; or we abductively infer that there must have been some pre-established harmony between the keys and our doors at home. Mark Steiner argues for the second option, claiming that we should accept that we are living in a "user-friendly universe", i.e. a universe whose deep structure is somehow attuned to the workings of the human mind and hence to the products of mathematical reasoning (Steiner 1998). Steiner's "anthropocentrism" is reminiscent of the old rationalist idea that it is part of God's creation to have made the world and our minds to fit like hand to glove. Hence, if, time and time again, mathematical thinking proves to be the royal road to physical knowledge about the world, then this should be taken to suggest that the deep structure of physical reality is of a kind that makes it amenable to description by mathematical reasoning.[3]

1.3 An Alternative Approach

The aim of the previous section was not to give a complete exposition of the various attempts to find a solution of the applicability problem. The aim was rather to indicate that *the* applicability problem does not exist: Depending on the philosophical background assumptions one accepts, there are different ways of conceiving the relationship between mathematics and physics. And depending on the version of the distinctness thesis one accepts, different versions of the applicability problem arise. While some may think of the applicability problem as a metaphysical issue that concerns the coordination between two categorically distinct ontological regions, others may focus on the methodological question of why a cognitive practice that is driven by genuinely inner-mathematical considerations proves successful in physics.

[3] It should be noted for the sake of completeness that there have been skeptical voices as well, effectively denying the distinctness thesis and, consequently, the existence of the applicability problem. The argument, in a nutshell, is this: If mathematics is really just a game-like invention, and if, furthermore, its inventors had genuinely physical purposes in mind, then there is nothing mysterious about the usefulness of mathematics in physical research. This view can easily be substantiated by several well-known examples from science history: Leibniz and Newton invented differential and integral calculus for the explicit purpose to describe systems with trajectories through space and time with forces acting on them. Given this practical background and given the ingenuity of its inventors, the applicability of differential and integral calculus is no more surprising than the fact that hammers are well-suited to drive nails. However, such a deflationary stance toward the applicability problem faces several problems: First, it would be a serious mistake to reduce the role of mathematics to that of a convenient tool for the successful framing of physical descriptions. Quite the opposite: In the face of lacking empirical data, physicists quite often turn to mathematics itself in order to discover novel theories or even previously unknown physical phenomena (cf., e.g., Steiner 1989, 1998; Colyvan 2001). Second, and closely related, it is also not true that the mathematical tools that proved useful in physics were always developed with genuinely physical purposes in mind. Some of the most productive mathematical innovations such as complex numbers, non-Euclidean geometries or spinors were regarded as purely theoretical first and went on to demonstrate their high practical relevance decades, sometimes even centuries later.

However, there is one characteristic that is common to all versions of the applicability problem as well as to their attempted solutions: Accepting the success of physics, and building on certain views concerning the nature of mathematics and physics, philosophers consider the applicability of mathematics a phenomenon that requires a philosophical *explanation*. Hence, philosophers pondering over (some version of the) applicability problem are generally in the business of finding an answer to a specific *why-question*—the question as to why mathematics can play the role it does in modern physics.

At first glance, it may seem natural enough to view the applicability of mathematics as a phenomenon that prompts an explanation-seeking why-question. After all, aren't philosophy and the sciences continuous in the sense that both seek to identify sufficiently interesting phenomena for which they then account by means of theoretical explanations? As natural as this view may seem, phenomenologists have traditionally been skeptical of this explanatory paradigm in philosophy. To be sure, the point is not that it is impermissible under any circumstances to construct philosophical theories for particular explanatory purposes. The point is rather that there is a tendency among philosophers to jump to explanations too quickly, thereby ignoring the fact that what is considered to be the explanandum is oftentimes contingent upon presuppositions that are not properly grounded in a faithful and unbiased description of the *things themselves*. Phenomenologically construed, it is only if we abstain from immediately jumping to explanations that we can prevent the risk of engaging in what Husserl has contemptuously called "standpoint philosophy" (*Standpunkt-philosophie*): Instead of forcing problems into a particular (and potentially artificial) theoretical mold, phenomenologists are driven by a deep respect for the *phenomena*, i.e., the things exactly as they are given in experience. On a phenomenological view, many philosophical problems could be solved—or even better: made to evaporate—if we resisted the temptation to interfere with pre-established metaphysical, ontological or epistemological schemes and put more effort in a faithful description of the phenomena.

In what follows, we will approach the applicability problem from a phenomenological point of view. This means, first, to shift attention away from the *why-question* and focus on the *how-question* instead. Hence, our aim will not be to explain a problem that, already at the level of its formulation, is contaminated with certain philosophical preconceptions about the nature of mathematics and physics. Rather, utilizing Husserl's idea of "an epoché in regard to all objective theoretical interests" (Husserl 1970, 135; Wiltsche 2012, 126-127), ready-made ontologies and epistemologies of mathematics and science will be, to use Husserl's apt term, *bracketed* in order to arrive at a more faithful and unbiased understanding of how mathematics is actually applied in the physical sciences.

Yet, approaching the applicability problem from a phenomenological viewpoint also means, second, to take seriously that mathematics is always applied by a community of historically, culturally and bodily situated subjects. As trivial as this may seem at first sight, it is a fact which, once recognized, opens up two basic avenues for inquiry: Addressing the how-question from a first-person perspective puts us in a position to gain a firmer grasp of the intentional structures that are operative in concrete cases of mathematical-physical theorizing. Drawing from the rich resources of phenomenological investigations of human consciousness, we will seek to identify and describe the sorts of intentional acts by means of which mathematical objects become applied in present-day physical research. However, while unveiling the intentional accomplishments that underlie contemporary physics is without doubt an important task, this *synchronic* view must be supplemented by a *diachronic* investigation into the *genetic origin* of the very idea of an amalgamation between mathematics and physics. It is very natural for us today to take the possibility of mathematized physics simply for granted. However, as the late Husserl of the *Crisis* was at pains to show, the early 17th century attempts of a complete mathematization of empirical reality mark a fundamental turning point in intellectual history whose philosophical consequences are not sufficiently understood until the present day. In our view, and for reasons that will become clear, the how-question can only be answered in a satisfactory manner if it is approached from a synchronic as well as from a diachronic perspective, and if ungrounded metaphysical and epistemological preconceptions about mathematics and physics are held to a minimum.

1.4 A Diachronic Investigation Into the Origin of Modern Physics

As the section title suggests, we begin by focusing on the case of modern physics as opposed to other empirical sciences and Aristotelean physics. The goal is to reconstruct or do a genetic investigation of the birth of modern physics: we ask how physics as a mathematized science became possible in the first place.

Aristotelean physics, if you allow the title, was not mathematized. It was Galileo (of course, not single-handedly) who gave birth to physics as a mathematized science. We admit that mathematics can be applied to Aristotelian physics in the same way that logic can be applied to philosophy. But mathematization and application are two distinct issues, as will become clear in the following. Simply put, one can do Aristotelean physics (and that's how it was done for centuries) without ever using mathematics while the same is not possible in the case of modern physics.

How, exactly, is mathematics used in modern physics? To be sure, there is the instrumental use: once we have the mathematical formulation of laws, the observables and the initial conditions, we can use mathematics to make calculations, predictions and inferences. But this role of mathematics is no different from its use in everyday life: If Arezoo puts five apples in the basket, and Harald adds another two, the grand total of apples can be determined arithmetically. But this is not the role of mathematics that is particularly "mysterious": Even though we might be deeply impressed by the effectiveness with which mathematics allows us to draw inferences from known facts or pre-established theories, the ability to do so is not significantly more mysterious than the general human capacity to use the powers of reason to go beyond the immediately given.

In modern physics, however, the relation between mathematics and the objects under consideration is typically much more intimate, especially at the level of fundamental laws. Whereas apples in a basket are accessible independently from the mathematical tools we might apply to them, any attempt to separate Newton's motion, Einstein's curvature of space-time or Schrödinger's wave function from the mathematical formalisms in which they are couched is doomed to failure. We take this as a first indication of what it means to think of modern physics as a mathematized science: Instead of just being applied to objects that could also be ascertained otherwise, mathematics seems to be crucial for the very *constitution* of the objects modern physics purports to describe. Hence, as a first approximation we might say that modern physics is a mathematized science in the sense that, at its core, it deals with idealized, exact objects—objects that are nowhere to be found in our ordinary experience of the world.

Let's begin with Galileo as the initiator of this kind of mathematized physics. The issue here is not Galileo's name, nor his individual contribution or the context of his work. The focus is on a rational reconstruction of a process: the mathematization of physics which is philosophically illuminating to us. Our interest in Galileo is thus similar to that of philosophically-minded historians such as Maurice Clavelin who rightly remarked that "Galilean science was first of all a transition from one conceptual framework to another, the replacement of one explanatory ideal with another and an unprecedented fusion of reason and reality" (Clavelin 1974, xi).[4]

Galileo suggested that nature is written in the language of Euclidean geometry. He regarded nature as composed not of objects of ordinary experience, but of fundamentally different objects such as triangles and circles. Of course he did not discover this as a fact about nature. Galileo mainly relied on theological arguments[5] in order to justify the appropriation of the method of Euclidean and Archimedean proportional geometry for the study of nature. The effect of this methodological innovation was a science whose objects were, at least in part, radically different from the objects we encounter under normal lifeworldly circumstances.

While in his workshop, Galileo worked with surfaces that could be polished in order to increase their smoothness. His remarkable innovation was to proceed to the limit of this process in the imagination. As surfaces can be thought of getting smoother and smoother, he declared, there must also be a perfectly smooth surface: a surface with no friction. Of course, such a surface cannot exist in the real world, just as the figures of Euclid—lines, triangles or circles—could not. They too are constructed as the ideal limit of a process: A line with no thickness is an ideality, the limiting pole of a sequence of lines with lesser and lesser thickness. The truly revolutionary aspect of Galilean science is to elevate such constructed idealities to the principle means through which all of reality must be studied. One of the many

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[4] To be sure, this is also the approach that is taken in the famous section 9 of Husserl's *Crisis* where the name "Galileo [...] is the exemplary index of an attitude and a moment, rather than a proper name" (Derrida 1989, 35; Husserl 1970, 57).

[5] Galileo's argument, in a nutshell, is that rigorous mathematical proofs allow us to participate in the perfection of God's knowledge. Hence, when an empirical problem can be dealt with mathematically, Galileo feels warranted to regard the geometrical model of the empirical target system as a truthful representation of how God perceives reality (cf., e.g., Galilei 1967, 103; McTighe 1967; Redondi 1998).

telling examples in Galileo's oeuvre is his treatment of projectile motion (cf., for the following, Wiltsche 2016, sec. 4): In order to "prove" that projectiles follow a semi-parabolic path, Galileo introduces a scenario which is built up from geometrical objects such as frictionless planes and perfectly spherical projectiles, and in which, consequently, no energy is lost due to friction or perturbation effects, in which objects are not attracted by a common center of gravity and in which the surface of the earth is treated as an Euclidean plane. Although Galileo's "proof" crucially depends on these counterfactual conditions, it is the idealized scenario that becomes prescriptive for our everyday experience: This is because, according to Galileo, the idealized scenario is nothing less than a truthful representation of how projectile motion would *really* look like if all causal impediments and accidents could be put aside. Hence, it is only through idealized objects such as spheres, planes or lines that we can catch a glimpse of the deep structure of reality.

As the late Husserl suggests, Galileo's methodological innovation inaugurated a process in which communities of scientists throughout history replace aspects of the natural surrounding world with mathematical objects. As a result of this process, the world of the modern mathematical sciences is not simply a continuation of ordinary experience and common sense, but rather a radical break. As we will see in more detail later, most of the objects with which we are concerned in physics are constituted very differently from the objects we encounter under normal, lifeworldly conditions. If this is correct, the question naturally arises: What does it mean for our understanding of science and reality that the former seems to proceed through a continuing transformation of the latter?

Galileo himself answered this question in a rather straightforward manner by raising his scientific method (an epistemic achievement) to the status of a fact about nature (an ontological truth). Although, of course, he did not deny that our surrounding world does not appear mathematical at all to us, Galileo argued that the perceived imperfections and irregularities do not belong to reality itself, but rather to our perception of it. It is only because of the limitations of our senses[6] that we fail to experience reality as it really is: perfect, unchanging and simple. In effect, Galileo thus extended what the ancients had assumed about the heavens to be true of the world in its entirety. While the ancients had restricted the applicability of mathematics to bodies with ethereal composition, Galileo went beyond the dualism

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[6] It should be noted that there are actually two distinct metaphysical arguments that operate in the background of Galilean physics. First, there is the doctrine of primary and secondary qualities that Galileo introduced in *Il Saggiatore* and that became common currency in philosophical circles through the works of Descartes, Locke, Hume and others. The second argumentative strategy, which seems to play a particularly prominent role in Galileo's scientific practice, is based on the distinction between *natural occurrences* and *phenomena* (cf. Koertge 1977; McAllister 1996): Natural occurrences are the physical processes, exactly as they occur under normal, lifeworldly conditions. Phenomena, on the other hand, are the abstract invariant forms that allegedly underlie natural occurrences. According to Galileo, a natural occurrence is always the result of one or more phenomena and great number of accidents. And although Galileo acknowledges that the accidents are responsible for the huge variety of observable natural occurrences, he claims that they must be systematically excluded from physics through the method of geometrical idealization.

of superlunar and sublunar by degrading our surrounding world to a mere veil behind which the real world of mathematical objects is hidden.

Of course, Galileo had no way of knowing that the hidden reality behind the veil of perception is mathematical in nature. And in light of the poor observational-predictive record of early Galilean mechanics,[7] he could not have substantiated this metaphysical assumption in the way it is justified nowadays, i.e. by baptizing it as the best explanation for the success of mathematized physics. What Galileo did, then, was to present as a discovery what was actually a bold methodological conjecture: that the very method Arab/Persian astronomers had used to study the heavenly bodies is applicable to nature in its entirety.

Galileo's systematic replacement of objects of ordinary experience with geometric figures was continued and radicalized in the works of Descartes, Leibniz, Newton and others. Of course, since he simply inherited Euclidean and Archimedean proportional geometry from the tradition before him, Galileo did not have the right tools for his ambitious program of a complete mathematization of nature. It was particularly Newton and Leibniz who took more radical steps by developing their own mathematical machinery which allowed for the idealization of interaction and the replacement of natural motion with differential equations. Utilizing this method, nature turned into a dynamic system radically different from the static order Galileo had envisioned. It is this "dialectical movement" of mathematical innovation and resulting re-conceptualizations of nature that, in our view, is a defining characteristic of the history of physics: Just as Newton replaced the tools Galileo had used, Einstein paved the way for replacing calculus and differential equations with group theory, thus assigning a more fundamental status to symmetries than to dynamical laws.

Let us conclude this sketch of a diachronic investigation of the genetic origin of modern physics by summarizing the most important results: The objects of modern mathematized physics are not adopted from the world of everyday experience, but are constituted in a fundamentally different way. In physics we do not study motion as it appears to us although the physical notion has its origin in the "intuitive sense of things moving". What we study in physics is rather motion as an idealized entity that is already mathematized—if you are a Newtonian, motion is constituted through

[7] Although it is not an easy task to determine the empirical adequacy of Galilean science from a contemporary perspective, the following episode shows how hard it was to apply Galilean mechanics successfully in the 17th century: Four years after Galileo's death the gunner Giovanni Ranieri attempted to apply Galileo's theory of projectile motion to his craft. However, as Ranieri reports in a letter to Evangelista Torricelli—Galileo's successor at the University of Pisa—the experimental results did not even come close to matching the theoretical predictions. Ranieri replicated one of Galileo's geometrical "proofs" by using an elevated gun to perform a number of point-blank shots. While the theory predicted a range of approximately 96 paces, Ranieri achieved ranges of 400 paces and more (cf. Segre 1991, 94-97). Particularly interesting is how Torricelli reacted to Ranieri's complaint: Torricelli explained the empirical inadequacy of Galileo's theory by pointing out "that Galileo [speaks] the language of geometry and [is] not bound by any empirical result" (Segre 1991, 44). Even more interesting is the fact that Galileo himself was perfectly aware of the practical insufficiencies of his own theory. Shortly after he has presented his "proof" that projectiles describe a semi-parabolic path, he freely admits that the "conclusions proved in the abstract will be different when applied in the concrete and will be fallacious to this extent, that neither will the horizontal motion be uniform, nor the path of the projectile a parabola" (Galilei 1954, 251).

differential equations; if viewed from the perspective of relativity theory, motion is constituted through modified equations with Lorentz factor. The possibility of being intentionally directed toward the world in this peculiar manner depends on two fundamental presuppositions: First, it depends on the existence of pure mathematics as the systematic study of idealities such as numbers, lines or polynomials. These idealities are constructed out of intuitively given objects of the life-world whose properties are characterized by a fundamental vagueness and imperfection: While the technologically mediated process of, say, decreasing the thickness or increasing the flatness of life-world objects is essentially open-ended, the true intellectual accomplishment behind pure mathematics lies in the ability of the human mind to jump to the ideal end-point of such empirically interminable processes and to study the resulting ideal "limit-shapes" independently from the concrete particulars that give rise to them. But of course, the mere existence of pure mathematics is not yet sufficient to view reality in a mathematized way. In order for a cognitive agent to immerse herself into the scientific image, as it is displayed in various physical theories, the human mind must, second, also be able to "apply" pure mathematics to the experiential world in a very specific and intimate way. Or, to put it differently: Viewing nature in a mathematized manner is the result of a quite peculiar process of constitution which essentially involves mathematics and which can be further explicated phenomenologically. And to this we will now turn.

1.5 A Synchronic Investigation Into the Constitution of the Objects of Physics

We have argued so far that, first, the objects in physics are constituted differently than the objects of everyday experience and that, second, the physical sciences proceed through a continuing replacement of aspects of our natural surrounding world with mathematical idealities. Given how crucial these two interrelated claims are for our overall argument, it is of utmost importance to be as clear as possible about their implications and philosophical underpinnings. Hence, in this section we will supplement the diachronic investigation from the previous section with a synchronic analysis of the intentional structures that underlie different kinds of cognitive involvement with reality. The primary aim of doing so is to explicate the key notions of *constitution* and *replacement*.

One of the most fundamental insights of phenomenology is that the objects of cognition—in science as well as in everyday life—are not simply given. That objects are given to us is rather a phenomenon that is itself in need of further clarification. What phenomenology seeks to offer, then, is a faithful description of the structures of consciousness that are operative when different kinds of objects are intended through different kinds of intentional acts. Since, phenomenologically construed, these structures are the very condition of the possibility of any directedness towards the world, it is only on the basis of a comprehensive description of intentionality that human cognition, its limits and its potential, can be properly understood.

Let us begin with a simple example from the perceptual realm: Arezoo is supposed to meet Harald at the Double R Diner for coffee and pie. While driving down Main Street, she sees a building some distance away. After pulling into the parking lot she recognizes the building's exterior from a photograph she has once seen: she has reached her destination and Harald is already waving from the inside.

If we adopt a reflective stance toward this perceptual episode, several interesting observations can be made. To begin with, there is a describable difference between what is meant in an act of perception and what is actually sensuously given. During her ride down Main Street, Arezoo is visually attending to a material object that she intends as a three-dimensional thing in space. Yet, if we focus on what is sensuously given, it is clear that the whole object is never visually available to her at once: All that is visually present at any given point in time is a two-dimensional appearance, i.e. a profile of the thing, as it appears from one particular perspective.

Furthermore, the way Arezoo experiences the sequence of two-dimensional appearances is highly structured: As Arezoo drives down Main Street and thus changes the vantage point from which the object is seen, she brings new two-dimensional appearances into view. And although this is hardly ever noticed in the usual course of events, these new two-dimensional appearances fulfill (or frustrate) Arezoo's anticipations of how the object will continue to appear in further acts of perception. For example, in intending the object as a three-dimensional thing in space, Arezoo has the anticipation that there is more to the thing than is revealed in one single glance and that, consequently, new profiles and features will enter her visual field when she changes her position. Anticipations of this kind are not a matter of inferential belief or judgment over and above the experience in which things are perspectivally given; they are rather an essential part of any such experience and hence part of what it means to experience an object as a three-dimensional thing in space.

Finally, there is an intimate relationship between how an object is intended and the structure of the anticipations that are co-given with the sensuous data. For instance, when Arezoo first spots the building from afar, the structure of her anticipations concerning further possible experiences is relatively indeterminate and open-ended: Since, at this stage, she intends the object just as "a building", she wouldn't be too surprised if further experiences revealed a sign that reads "Fire Department". After recognizing the building and perceiving it as what it actually is, however, the structure of her anticipations is much more narrow and specific to what she knows about the Double R Diner.

As these analyses are supposed to show, perception is not merely, or even chiefly, about what is actual. Rather, the sensuously given is always and necessarily embedded into an open, but structured manifold of anticipations concerning further possible experiences. While the structure of these anticipations is what phenomenologists call the *horizon of experience*, the rule that governs the structure of the horizon is called the *sense* or the *noema* of an experience. So, when Arezoo first spots the object from afar, she does so by intending it through a noema that could be linguistically expressed by the term "building". It is this noema that then awakens a structured horizon of possible further experiences against the background of which new sensory data is constantly projected. One can think of the horizon in terms of a *space of* *possibilities* that plays an instrumental role in the evaluation of perceptual episodes: Whenever Arezoo changes her bodily location and thus receives new sensory input, this input must be harmonized with what is prescribed through the horizon. If the harmonization succeeds, i.e. if the sensuously given is compatible with the possibilities that are laid out in the horizon, then Arezoo's perceptual encounter with the intended thing proves successful. If, on the other hand, the harmonization fails—if, for instance, the building turns out to be an ingeniously designed hologram with no backside at all—, then Arezoo will have to accept that the noema through which the object has been intended must be revised. It is this process of intending objects through specific noemata and then constantly projecting new sensory data against horizons of possible further experiences that phenomenologists call *constitution*. Of particular importance in this context are those aspects of experience that remain invariant across a sequence of changing appearances. When Arezoo perceives the building in front of her from different viewpoints, many aspects of her experience are variable and in a constant state of flux: for instance, the perceived shape of the rectangular building will always be different, depending on the viewpoint from which the building is seen. However, what remains invariant over all varying shape-appearances is, for instance, the lawful angular relations between the perceiving subject on the one hand and each of the sides of the perceived thing on the other. It is invariances of this kind on which the constitution of *perceptual objectivity* is ultimately founded.

It is hard to overestimate the importance of the horizonal structure of intentionality for our understanding of human cognition in general and the notion of constitution in particular. Instead of thinking of intentionality in terms of a static, one-way relation between act and object, intentional directedness turns out to be a dynamic process in which objects are constituted by projecting ever-changing appearances against a horizon of possible further experiences. What makes this insight even more relevant is that the horizonal structure is not just found in perceptual experience. An example will help to illustrate how a phenomenological framework sheds light on the nature of scientific constitution.[8]

Imagine an experimental setup in which an EF-probe is used to measure the strength of an electric field at various points between two charged conductors. Imagine furthermore that two persons, Audrey and Dale, are invited to follow the experiment and describe what they are experiencing. Dale, a complete layman in physics, reports that he is observing a yellow-black piece of electronics whose display shows different digits, depending on where the piece of electronics is put. Audrey, on the other hand, has a PhD in physics and offers a rather different description. She knows that the charged conductors create an electric field that permeates the space between the conductors; she knows that the field exerts a certain force on the EF-probe; and she also knows that the strength of the force acting on the probe depends on two factors, the charge of the probe and the strength of the electric field. Most importantly, however, Audrey is in possession of a mathematical model that allows her to give a quantitative determination of the relationship between the strength of

[8] The following example is a modification of an example found in Weyl (1948, 393-397; 1949, 113-114).

the field, the charge of the probe and the force that acts on the probe. Building on this background knowledge, Audrey is able to describe the situation in front of her as what it actually is: an experimental setup in which two conductors create an electric field whose strength at various points is measured by the EF-probe.

In light of the fixation on propositional knowledge that is still widespread in contemporary philosophy, it might be tempting to explain the differences between Dale's and Audrey's descriptions solely in terms of the background knowledge upon which they draw. However, although there is no point in denying that the available background knowledge does matter in the example at hand, it would be a serious oversimplification to reduce the difference between Audrey and Dale to the differences between what they *know* about physics. Phenomenologically construed, a more complete picture only emerges if we take into consideration how different stocks of background knowledge are used to intend one and the same situation in fundamentally different manners.

As we have seen, Dale intends the experimental setup through a noema with which he is familiar from the context of everyday practice. By intending the EF-probe as a "yellow-black piece of electronics", Dale generates a plethora of more or less determinate anticipations concerning various physical features of the thing. For instance, Dale will have the implicit anticipation that the piece of electronics will reveal a momentarily hidden backside if it is turned around, or that its size will remain the same if it is moved from A to B. Yet, at the same time, other aspects are left unspecified: Since Dale's lack of knowledge about physics only allows him to intend the probe as a normal material thing without special scientific significance, the information it produces lies beyond the scope of his attention. Dale wouldn't be surprised in the least if the numerical values on the probe's display would change erratically, or if the display didn't show any digits at all.

Although they are located in close spatiotemporal proximity, and although they seem to visually attend to the same scenario, the *how* of Audrey's intentional directedness is significantly different from the manner in which Dale intends the situation in front of him. This becomes apparent from Audrey's description of what she is experiencing: The fact that she is able to describe the scenario as what it actually is shows that she understands the scientific significance of the experimental setup. However, in order for this kind of scientific understanding to occur, it is not necessary—and, in fact, not even pertinent—to intend the EF-probe as a material thing that will reveal a backside when turned around or whose size will not be affected if it is moved from A to B. In the same sense in which we can "look through" a freehand drawing of a circle and intend an ideal geometrical circle instead, Audrey strips the probe of all its sensible properties such as color or texture, and intends it as a geometrical point with which a scalar factor and a vector quantity are associated. This *shift of attitude* is the result of intending the probe through a very specific noema, namely through the ideal mathematical content "$\mathbf{F}(P) = e \cdot \mathbf{E}(P)$".

What object one experiences is always underdetermined by the experiential data that is available at any given point in time. In the example at hand, the EF-probe could be constituted as a yellow-black piece of electronics, as an aesthetically appealing piece of art, as a paperweight or as a point-like, but otherwise unspecified carrier of certain numerical values. Which object one experiences depends on the noema through which the how of the intentional relation between subject and world is specified. In Audrey's case, it is due to the noema "$\mathbf{F}(P) = e \cdot \mathbf{E}(P)$" that she is able to "look through" the materiality of the probe and to intend it as a geometrical point with which a scalar factor and a vector quantity are associated. At the same time, the noema also awakens a rigidly structured horizon that determines the relations between the probe's charge e, the measured vector force $\mathbf{F}$ and the field strength $\mathbf{E}$ in an unambiguous, precise and mathematically rigorous way. Whenever Audrey receives new data by varying the position of the probe, this new data must be harmonized with what is prescribed through the horizon. If the harmonization succeeds, i.e. if the data is compatible with the space of possibilities that is determined by the noema, then Audrey's encounter with the situation in front of her proves successful. If, on the other hand, the harmonization fails—if, for instance, all properties of the experimental setup are held constant, but the value of the measured vector force $\mathbf{F}$ changes nevertheless—, then Audrey will have to accept that the noema through which the scenario has been intended must be revised.

As the results of our synchronic investigation suggest, there is an important sense in which constitution in science and constitution in everyday contexts are structurally analogous. Being intentionally directed towards reality always means to intend the objects around us through a noema that awakens a more or less structured horizon of possible further experiences. One can think of the horizon as a space of possibilities that is instrumental in the evaluation of any encounter with the world: For whether such an encounter is deemed successful depends on whether the experiencing subject succeeds in harmonizing new incoming experiential data with what is prescribed through the horizon. Note, however, that this harmonization is essentially processual in character and also requires an active role on the part of the observer: The experience of an object as objectively existing is never founded on one isolated perspectival encounter with the object. In order to penetrate the object's full ontological depth, the experiencing subject must "probe" the horizon by constantly gaining new experiential data that can then be projected against the horizon. In simple perceptual cases, new sensuous data is gained through kinesthetic movement, i.e. by varying the location of the observer's body. In the earlier example of the constitution of an electric field, the horizon is explored by varying the location of the EF-probe, the latter serving as a technological extension of Audrey's body.

There is an obvious question that arises at this point: One of the key insights that have emerged from the previous section was that objects in physics and objects of everyday experience are constituted differently. But how can this be the case if, as we have now claimed, there is a structural analogy between constitution in science and constitution in simple perceptual situations? To provide an answer to this question, it is necessary to distinguish clearly between descriptions of the formal (i.e. domain-independent) structures of intentionality and descriptions of the various ways in which intentionality is instantiated in particular domains of cognitive engagement.

Phenomenologically construed, the structural analogy between physical and everyday constitution stems from the fact that the dynamic interplay between noema, horizon and experiential data represents the very core of any intentional relation between subject and any kind of object. However, the differences between physical and everyday objects come into view once we pay closer attention to the noemata through which these objects are constituted in their respective domains. When we are intentionally directed toward individual objects in everyday situations, we always experience these objects as particular instances of more general *empirical types*. For example, in being directed towards the EF-probe through the noema "yellow-black piece of electronics", Dale experiences the intended object as an instantiation of the empirical types "yellow", "black" and "piece of electronics". Empirical types are bundles of anticipations that were formed over the course of previous experiences, and by drawing qualitative analogies between objects that are deemed similar. If empirical types are used to specify the manner in which particular objects are intended, they awake a horizon of further possible experiences. Depending on Dale's previous experiences with other pieces of electronics, he will anticipate, for example, that the intended object will reveal an array of wires and electric components if it is cracked open, or that its perceived shape will change in a specific, but only qualitatively determined manner if it is turned around. Yet, since empirical types are based on association and similarity, such if-then anticipations are characteristically vague and imprecise.

While the manner in which the experimental setup presents itself to Dale is characterized by a fundamental vagueness, Audrey experiences the scenario in a significantly different manner. To begin with, as we have seen, the noema "$\mathbf{F}(P) = e \cdot \mathbf{E}(P)$" functions like a filter, screening off all sensible properties like color or texture. Even though Audrey is still aware of the fact that she is intending a segment of the physical world, the scenario she is experiencing is devoid of the buzzing and blooming confusion of sensible qualities, which is characteristic of normal life-world experiences. The EF-probe is constituted as a geometrical point with only one intrinsic physical property, its having a charge with the numerical value e. Hence, since the probe is constituted as a mathematical ideality, and since mathematical idealities are constituted as remaining self-identically the same, comparisons with other point-like probes are not based on similarity and association, but on an objective ordering of the value of e. However, apart from stripping the intended scenario from its sensible properties, the ideal mathematical content "$\mathbf{F}(P) = e \cdot \mathbf{E}(P)$" also transforms the anticipations that are co-given with the available experiential data. Whereas Dale's anticipations are vague and imprecise, the space of possibilities awakened by the noema "$\mathbf{F}(P) = e \cdot \mathbf{E}(P)$" determines the correlations between all properties of the intended scenario in a mathematically rigorous and quantitatively precise manner. Hence, it is Audrey's ability to intend the experimental scenario through a specific, non-morphological noema that makes reality amenable to a quantitative, mathematically rigorous treatment. Audrey has not just constituted reality: By intending the experimental setup through an ideal, mathematical content, Audrey has engaged in a very peculiar process of constitution which, following Husserl, is called *mathematization*.

In his seminal *Galileo Studies*, Alexandre Koyré notes that "Galileo's [...] mental attitude [...] is not purely mathematical [but] *physico-mathematical*" and that, although "Galileo tells us to start from experience, [...] this 'experience' is not the raw experience of the senses" (Koyré 1978, 108). Not only do we agree; we also suggest that Koyré's remarks can only be fully understood if they are read against a phenomenological background:[9] As far as his work in mechanics and kinematics is concerned, Galileo's primary achievement was neither the invention of new instruments, styles of reasoning or experimental techniques. Nor does the novelty of his approach lie in the mere application of mathematics to empirical phenomena. In our view, what justifies the title "father of modern science" more than anything was Galileo's trailblazing innovation to *mathematize all of reality by intending it through ideal-mathematical noemata*. Paradoxically as it may sound, this revolutionary new way of constituting reality resulted both in an impoverishment and in an enrichment of the empirical world. As our earlier example shows, the segment of reality towards which Audrey is intentionally directed is impoverished in the sense that it is devoid of sensible properties such as color or texture. At the same time, however, Audrey's mathematized world is significantly richer than Dale's: Audrey understands the situation in front of her because she not only intends a point-like probe with which a scalar factor and a vector quantity are associated. Audrey also intends additional layers of reality that, while not accessible to Dale, are crucial for understanding the scientific significance of the experimental situation. As long as Audrey immerses herself into the scientific image of reality, the world she experiences is populated by point-like probes, numerical values, vector forces and fields. Following Galileo's footsteps, Audrey has *replaced* the life-world of everyday experience with a scientific image of reality by intending the world through the noema "$\mathbf{F}(P) = e \cdot \mathbf{E}(P)$".

Galileo's achievement of intending reality through mathematical noemata not only initiated the historical process of replacing more and more aspects of our natural surrounding world with increasingly sophisticated mathematical idealities. A crucial by-product of Galileo new scientific vision was also to make these mathematical idealities indispensable for the very definition of objectivity in physics. As we have seen earlier, Galileo did not think of his geometrical models in terms of willful distortions of reality that must later be de-idealized in order to account for the phenomena as they occur under normal life-world conditions. Galileo rather considered his idealized models to be the only way to catch a glimpse of how the deep-structure of reality objectively looks like. A consequence of this interpretation is that the mathematical idealities out of which Galileo's models are constructed become prescriptive for experience: If one accepts, as Galileo does, that the idealized model represents the objective being of reality, then life-world phenomena must be regarded as mere approximations to the ideal case which is nowhere to be found in the realm of everyday experience. Of course, from a contemporary per-

[9] The suggestion to read Koyré from a phenomenological perspective is by no means far-fetched: Not only was Koyré a student of Husserl in Göttingen; Koyré had plans to write his dissertation on the antinomies of set theory under Husserl's supervision. What is more, as Parker has argued in detail, there are good reasons to believe that Koyré's later interpretation of Galilean physics was heavily influenced by Husserl's take on the issue (cf. Parker 2017). Although the phenomenological traces in Koyré's oeuvre have been overlooked by many, there are, of course, exceptions. For instance, Michel Foucault remarks that "we run across phenomenology in someone like Koyré [...] who [...] developed a historical analysis of the forms of rationality and knowledge in a phenomenological perspective" (Foucault 1998, 438).

spective Galileo's identification of simple geometrical models with objective reality must appear somewhat naive. However, the important point is that determining the notion of objectivity solely by mathematical means is still essential to the practice of physical theorizing. Take, for instance, the principle of covariance that lies at the heart of classical and relativistic mechanics: By requiring that the form of the laws of nature must be preserved under transformation from one reference frame to the other, the concept of physical objectivity is solely defined in terms of mathematical transformation rules that specify which properties remain invariant within the allowable group of transformations (Cassirer 1953; Kosso 2003). Hence, the guiding ideal of objectivity turns out to be inseparable from mathematical idealities such as the Lorentz transformations.

1.6 Conclusion

Let us come to a final conclusion. Nature as the subject of modern mathematical science is mathematical because *we have made it so*. That is, the match between mathematics and physics is not a match made in heaven but a match made on earth, through a long and arduous process of mathematization and co-constitution of sciences alongside one another (in this case physics and mathematics). In focusing on the how-question, our aim was to highlight the epistemological aspect of the applicability problem and the role humans play in the ongoing project of the mathematization of nature. Their role is certainly not that of passive spectators of another world that is hidden behind the phenomena. Scientific agents are rather active participants who are constructing and re-constructing objects in order to mathematize different aspects of nature. Seen from this perspective, then, *God* is a mathematician of course, if one recognizes the human as a deity.

Yet, it is understandable how to a physicist like Wigner or Tegmark it might appear that mathematical physics represents the true and actual nature. The process of idealization is hidden from the eye of the scientist for several reasons. In textbooks the views of previous scientists are always cast in modern notation, and reformulated using the current understanding of science. If the history of science is mentioned at all, then its role is that of a confirmation of a cumulative image of science. This history of science for scientists, as Grattan Guinness rightly puts it, aims at portraying *a royal road to us* (cf. Grattan-Guinness 1990, 157).

While for the scientist it is convenient—if not necessary—to forget the origins of her own science, "the original formation of meaning", the philosopher is required to go back and peel away the layers of this already formed "onion" to see what is really inside. While the scientist or the mathematician takes the science of his or her time as a given, the philosopher questions this very science. Such critique and questioning is but the task of the philosophical mind. Otherwise, we too will fall in the trap of miracles and mysteries by forgetting the very origins of physics and the continuous acts of re-conceptualization. Thus, formulating the relationship between mathematics and physics as an *application* is a major source of the problem. Application of one area to another assumes a distinctness which needs to be bridged. In the case of modern theoretical physics, we, unlike other commentators, showed that there is no such distinctness: the objects of physics are constituted mathematically.

Now, this is the beginning of a pluralistic project in which it becomes possible to study the (dynamic) relationship(s) of mathematics with other empirical sciences such as biology, and with other non-empirical sciences such as mathematics itself. We can ask how the previously *unexpected* relationship between different areas of mathematics are possible, for instance, how analytic geometry or algebraic topology are possible. The solution, we conjecture, will arise as a result of the study of the constitution of the objects of these *mixed* fields. In the case of biology, however, particular reasons can be given as to why biological phenomena don't allow mathematization in the same way that objects of physics do (cf. Islami 2017).

Finally, we are perfectly aware that there are many open questions,[10] for example how our phenomenological approach to mathematized physics can account for the empirical adequacy of our most successful physical theories. How is it, the critic might ask, that mathematically constituted objects of physics appear in equations that successfully predict the precise value of quantities (with negligible error) in experiments? This is an important and elaborate question which requires the space of its own. Schematically put, the answer involves an account of what an experiment is, how something comes to be a quantity, what we mean by prediction etc. Should it turn out to be impossible to deal with these issues in a constructive way, our position runs the risk of collapsing into an extreme form of idealism. Moreover, and to complete our answer to the *applicability problem*, we need a phenomenological excavation into the origins of mathematics and how it has become the pure abstract mathematics of the 20th century. Instead of beginning with a readymade ontology and epistemology, we suggest that we study mathematics as used and practiced. Mathematics understood by Wigner was more or less formalist, and only representative of the pure mathematics of the 20th century. It was then this forgetting of one's own position in history that bred miracles and mysteries. To this ailment, phenomenology has a cure.[11]

References

Bueno, O. and French, S. (2018). *Applying Mathematics. Immersion, Inference, Interpretation*. Oxford University Press, Oxford.

Cassirer, E. (1953). *Substance and Function, and Einstein's Theory of Relativity*. Dover, New York.

Clavelin, M. (1974). *The Natural Philosophy of Galileo. Essay on the Origin and Formation of Classical Mechanics*. MIT Press, Cambridge.

[10] It should also be noted that space constraints will prevent us from commenting on two recent solutions to the applicability problem that are in some ways similar to our own approach, namely da Silva's transcendental-phenomenological account (da Silva 2017) and the account that has recently been developed by Bueno and French (2018).

[11] This is the subject of our future work.

Colyvan, M. (2001). The miracle of applied mathematics. *Synthese*, 127:265–78.
da Silva, J. J. (2017). *Mathematics and Its Applications. A Transcendental-Idealist Perspective*. Springer, Cham.
Dawid, R. (2013). *String Theory and the Scientific Method*. Cambridge University Press, Cambridge.
Derrida, J. (1989). *Edmund Husserl's Origin of Geometry: An Introduction*. University of Nebraska Press, Lincoln.
Dirac, P. A. M. (1939). The relation between mathematics and physics. *Proceedings of the Royal Society*, 59:122–129.
Foucault, M. (1998). *Aesthetics, Method, and Epistemology*. New Press, New York.
Galilei, G. (1954). *Dialogues concerning two new sciences*. Dover, New York.
Galilei, G. (1967). *Dialogue concerning the two chief world systems*. University of California Press, Berkeley and Los Angeles.
Gödel, K. (1983). Russell's mathematical logic. In Benacerraf, P. and Putnam, H., editors, *Philosophy of Mathematics*, pages 447–469. Cambridge University Press, Cambridge.
Grattan-Guinness, I. (1990). Does the history of science treat the history of science? the case of mathematics. *History of Science*, 28:149–173.
Gross, D. (1988). Physics and mathematics at the frontier. *Proceedings of the National Academy of Science of the United States of America*, 85:8371–8375.
Hilbert, D. (1992). *Natur und mathematisches Erkennen*. Birkhäuser, Basel.
Hossenfelder, S. (2018). *Lost in Math. How Beauty Leads Physics Astray*. Basic Books, New York.
Husserl, E. (1970). *The Crisis of European Sciences and Transcendental Phenomenology. An Introduction to Phenomenological Philosophy*. Northwestern University Press, Evanston.
Islami, A. (2017). A match not made in heaven: on the applicability of mathematics in physics. *Synthese*, 194(12):4839–4861.
Koertge, N. (1977). Galileo and the problem of accidents. *Journal in the History of Ideas*, 28(3):389–408.
Kosso, P. (2003). Symmetry, objectivity, and design. In Brading, K. and Castellani, E., editors, *Symmetries in Physics. Philosophical Reflections*, pages 413–424. Cambridge, Cambridge University Press.
Koyré, A. (1978). *Galileo Studies*. Harvester Press, Hassocks.
Lesk, A. M. (2000). The unreasonable effectiveness of mathematics in molecular biology. *The Mathematical Intelligencer*, 22/2:28–37.
McAllister, J. (1996). The evidential status of thought experiments in science. *Studies in History and Philosophy of Science*, 27(2):247–273.
McTighe, T. (1967). Galileo's Platonism: A Reconsideration. In McMullin, E., editor, *Galileo. Man of science*, pages 365–386. Basic Books, New York.
Parker, R. K. (2017). The History Between Koyré and Husserl. In Raffaele Pisano, J. A. and Drozdova, D., editors, *Hypotheses and Perspectives in the History and Philosophy of Science. Homage to Alexandre Koyré 1892-1964*, pages 243–275. Springer, Dordrecht.

Penrose, R. (2004). *The Road to Reality: A Complete Guide to the Laws of the Universe*. Vintage Books, London.

Redondi, P. (1998). From Galileo to Augustine. In Machamer, P., editor, *The Cambridge Companion to Galileo*, pages 175–210. Cambridge University Press, Cambridge.

Segre, M. (1991). *In the Wake of Galileo*. Rutgers University Press, New Brunswick.

Smolin, L. (2006). *The Trouble with Physics*. Houghton Mifflin, Boston.

Steiner, M. (1989). The application of mathematics to natural science. *Journal of Philosophy*, 86(9):449–480.

Steiner, M. (1998). *The Applicability of Mathematics as a Philosophical Problem*. Harvard University Press, Cambridge, MA.

Tegmark, M. (2008). The mathematical universe. *Foundations of Physics*, 38(2):101–150.

Velupillai, K. (2005). The unreasonable ineffectiveness of mathematics in economics. *Cambridge Journal of Economics*, 29:849–872.

Weyl, H. (1948). Wissenschaft als symbolische Konstruktion des Menschen. *Eranos-Jahrbuch*, 16:375–431.

Weyl, H. (1949). *Philosophy of Mathematics and Natural Science*. Princeton University Press, Princeton.

Wigner, E. (1960). The unreasonable effectiveness of mathematics in the natural sciences. *Communications in Pure and Applied Mathematics*, 13:1–14.

Wiltsche, H. A. (2012). What is Wrong with Husserl's Scientific Anti-Realism? *Inquiry*, 55(2):105–130.

Wiltsche, H. A. (2016). Mechanics Lost: Husserl's Galileo and Ihde's Telescope. *Husserl Studies*, 33(2):149–173.